Sydney Girls High School
2022
Trial Higher School Certificate
Examination

Mathematics Extension 2

General Instructions	• Reading time – 10 minutes • Working time – 3 hours • Write using a black pen • Calculators approved by NESA may be used • A reference sheet is provided • In Questions 11-16, show relevant mathematical reasoning and/or calculations
Total marks: 100	**Section I – 10 marks (pages 2-6)** • Attempt Questions 1-10 • Allow about 15 minutes for this section **Section II – 90 marks (pages 7-14)** • Attempt Questions 11-16 • Allow about 2 hour and 45 minutes for this section

Name: …………………………………………………… Teacher: ……………………………………………………	**THIS IS A TRIAL PAPER ONLY** It does not necessarily reflect the format or the content of the 2022 HSC Examination Paper in this subject.

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Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Consider the following statement for real numbers a and b:

'If $a < b$ then $a^3 < b^3$.'

Which of the following is the converse of this statement?

A. If $a < b$ then $a^3 \geqslant b^3$.

B. If $a \geqslant b$ then $a^3 \geqslant b^3$.

C. If $a^3 \geqslant b^3$ then $a \geqslant b$.

D. If $a^3 < b^3$ then $a < b$.

2 Which expression is equal to $\displaystyle\int \frac{1}{\sqrt{2-x^2}}\,dx$?

A. $\sin^{-1}\dfrac{x}{2} + C$

B. $\dfrac{1}{2}\sin^{-1}\dfrac{x}{2} + C$

C. $\sin^{-1}\dfrac{x}{\sqrt{2}} + C$

D. $\dfrac{1}{\sqrt{2}}\sin^{-1}\dfrac{x}{\sqrt{2}} + C$

3 A particle of mass 3 kg is subject to a force $F = \frac{1}{2x^2}$, where x is the particle's displacement and $x \neq 0$. If the particle's velocity is v, which of the following is a correct expression for v^2?

A. $v^2 = -\frac{1}{6x} + C$

B. $v^2 = -\frac{1}{3x} + C$

C. $v^2 = \frac{1}{6x} + C$

D. $v^2 = \frac{1}{3x} + C$

4 Which expression is equal to $\int_0^1 xe^{3x} dx$?

A. $6e^3 - 9$

B. $6e^3 + 9$

C. $\frac{1}{9}(2e^3 - 1)$

D. $\frac{1}{9}(2e^3 + 1)$

5 Let $z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

Which of the following is a correct expression for z^9?

A. $z^9 = 1$

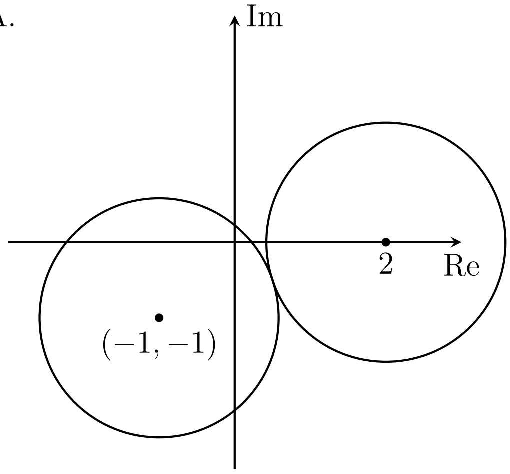
B. $z^9 = -1$

C. $z^9 = \frac{1}{512}$

D. $z^9 = -\frac{1}{512}$

6 Which diagram best represents the solution to the equation $|z+2| = |z-1-i|$?

A.



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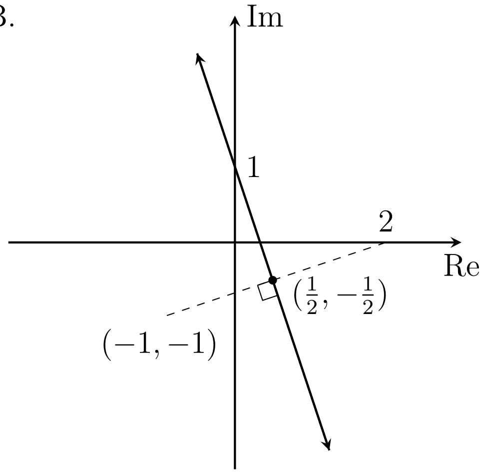

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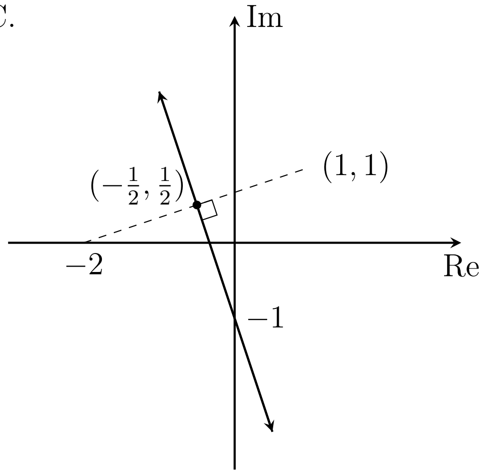
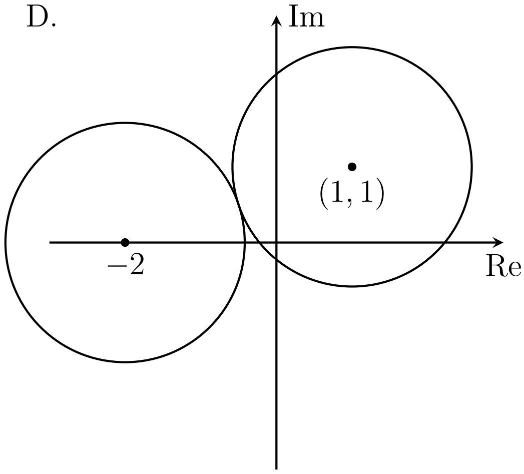
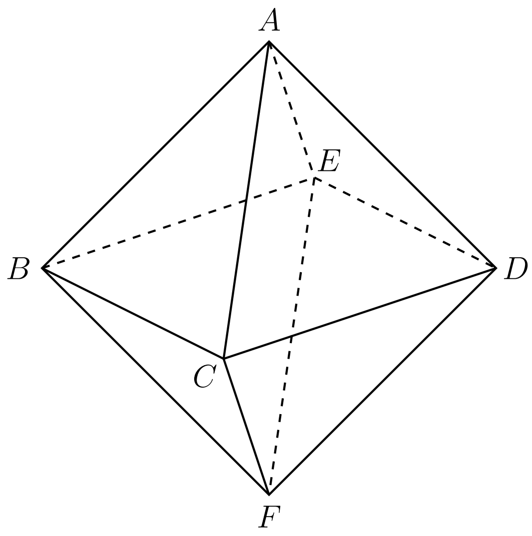


D.



7 In the diagram below, $ABCDEF$ is a regular octahedron.



Note that $BCDE$ is a square.

Which of the following vectors is equal to $\overrightarrow{BC}$?

A. $\overrightarrow{CA} + \overrightarrow{AB} + \overrightarrow{BD}$

B. $\overrightarrow{EA} + \overrightarrow{AF} + \overrightarrow{FD}$

C. $\overrightarrow{BE} + \overrightarrow{EA} + \overrightarrow{AD}$

D. $\overrightarrow{BF} + \overrightarrow{FA} + \overrightarrow{AE}$

8 Let $P(z) = z^3 + pz^2 + 25p$, where p is a real number.

Given that $1 + 2i$ is a zero of $P(z)$, which of the following is a real zero of $P(z)$?

A. $-\dfrac{5}{2}$

B. $\dfrac{5}{2}$

C. $-\dfrac{1}{3}$

D. $\dfrac{1}{3}$

9 Suppose that a, b, c and d are integers such that $a+b$ divides $ac-bd$.

Which of the following must be divisible by $a+b$?

A. $ab+cd$

B. $ab-cd$

C. $bc-ad$

D. $bc+ad$

10 Let $\underset{\sim}{a}$ and $\underset{\sim}{b}$ be non-zero vectors satisfying

$$(\underset{\sim}{a} \cdot \underset{\sim}{b})\,\underset{\sim}{a} = \left|\underset{\sim}{a}\right| \underset{\sim}{b}.$$

Which of the following is necessarily true?

A. $\text{proj}_{\underset{\sim}{b}}\,\underset{\sim}{a} = \dfrac{\underset{\sim}{b}}{\left|\underset{\sim}{a}\right|}$

B. $\underset{\sim}{a} = \pm\widehat{\underset{\sim}{b}}$

C. $\left|\underset{\sim}{b}\right| = 1$

D. $\left|\underset{\sim}{a} \cdot \underset{\sim}{b}\right| = \left|\underset{\sim}{a}\right|$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Start each question on a new sheet of paper.

Question 11 (15 marks) Use a new sheet of paper.

(a) Let $u = 5i$ and $v = -5 - i$. **2**

Find $\dfrac{\overline{u}}{iv}$, giving your answer in Cartesian form.

(b) (i) Find the square roots of $7 + 24i$. **2**

(ii) Hence, solve $2z^2 - \sqrt{7}z - 3i = 0$. **2**

(c) Find $\displaystyle\int \frac{1}{1 + e^{-x}}\, dx$. **2**

(d) Use an appropriate substitution to find $\displaystyle\int \frac{1}{(1 + x^2)^{\frac{3}{2}}}\, dx$. **3**

(e) Find a and b if $\begin{bmatrix} 1 \\ a \\ b \end{bmatrix}$ is perpendicular to both $\begin{bmatrix} -4 \\ 4 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ 3 \\ 2 \end{bmatrix}$. **2**

(f) Find p if the vectors $\begin{bmatrix} p \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} p \\ 0 \\ 2 \end{bmatrix}$ meet at an angle of 60°. **2**

End of Question 11

Question 12 (15 marks) Use a new sheet of paper.

(a) Evaluate $\displaystyle\int_0^1 \frac{x+3}{\sqrt{4-2x-x^2}}\,dx$. **3**

(b) Find $\displaystyle\int \sin^4 x \cos^5 x\,dx$. **3**

(c) Consider the points $A(-1,-2,4)$, $B(3,5,3)$ and $P(0,-1,-4)$.

(i) Find $\overrightarrow{AP}$ and $\overrightarrow{AB}$. **1**

(ii) Find $\text{proj}_{\underset{\sim}{b}}\,\underset{\sim}{p}$, where $\underset{\sim}{p} = \overrightarrow{AP}$ and $\underset{\sim}{b} = \overrightarrow{AB}$. **2**

(iii) Hence, find the distance from the point P to the line AB. **2**

Give your answer correct to four decimal places.

(d) (i) Use Euler's formula to show that $\sin\theta = \dfrac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$. **2**

(ii) Hence, show that $\sin^3\theta = \dfrac{1}{4}(3\sin\theta - \sin 3\theta)$. **2**

End of Question 12

Question 13 (14 marks) Use a new sheet of paper.

(a) Evaluate $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{1-\cos x}\, dx$. **3**

(b) Sketch the region on the Argand diagram defined by $|z-2i| > \text{Im } z$. **3**

(c) The height h cm of a horse on a merry-go-round after t seconds is given by

$$h = a\cos nt + c.$$

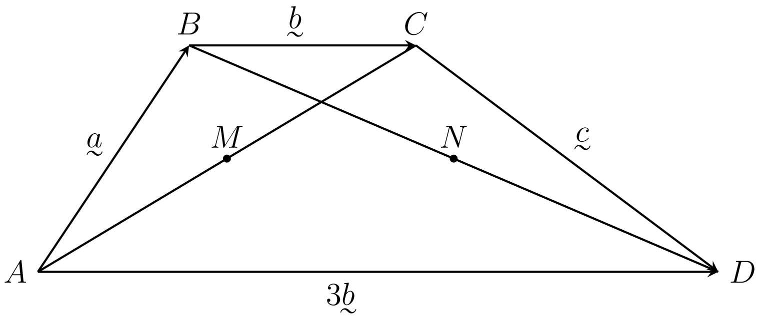
Initially, the horse is at its lowest point 40 cm above the ground.

Three seconds later, the horse reaches its highest point for the first time 110 cm above the ground.

(i) Determine the values of a, n and c. **2**

(ii) For how long during the first 6 seconds is the horse more than 1 metre above the ground? Give your answer correct to two decimal places. **2**

(d) Let $ABCD$ be a trapezium with $\overrightarrow{AB} = \underset{\sim}{a}$, $\overrightarrow{BC} = \underset{\sim}{b}$, $\overrightarrow{CD} = \underset{\sim}{c}$ and $\overrightarrow{AD} = 3\underset{\sim}{b}$, as shown below. Let M and N be the midpoints of the diagonals AC and BD respectively.



(i) Find expressions for $\overrightarrow{AM}$ and $\overrightarrow{AN}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$. **2**

(ii) Hence, show that $\overrightarrow{MN} = \underset{\sim}{b}$. **2**

End of Question 13

Question 14 (15 marks) Use a new sheet of paper.

(a) Let S be the sphere with centre the origin and equation $x^2 + y^2 + z^2 = 10$.

Let ℓ be the line with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

(i) Find A and B, the points of intersection of S and ℓ. **2**

(ii) Find $\angle AOB$ to the nearest degree. **2**

(b) A particle moves in simple harmonic motion such that $\ddot{x} = -9(x - 7)$, where x is the displacement of the particle from the origin in metres. The particle is initially at the origin with velocity, v, equal to 28 ms^{-1}.

(i) Express v^2 as a function of x. **2**

(ii) Hence, find the particle's maximum speed. **2**

(c) (i) By considering the binomial expansion of $(\cos\theta + i\sin\theta)^5$, show that **2**

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

(ii) Show that $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$. **2**

(iii) Show that $x = \sin\frac{\pi}{10}$, $x = \sin\left(-\frac{3\pi}{10}\right)$ and $x = 1$ are roots of **2**

$$16x^5 - 20x^3 + 5x - 1 = 0.$$

(iv) Given that $x = \sin\frac{\pi}{10}$ and $x = \sin\left(-\frac{3\pi}{10}\right)$ are double roots of $16x^5 - 20x^3 + 5x - 1 = 0$, show that **1**

$$\sin\tfrac{\pi}{10}\sin\tfrac{3\pi}{10} = \tfrac{1}{4}.$$

End of Question 14

Question 15 (15 marks) Use a new sheet of paper.

(a) It is given that

$$(x^3 + 8) = (x + 2)(x^2 - 2x + 4).$$

(i) Find real numbers A and B such that **2**

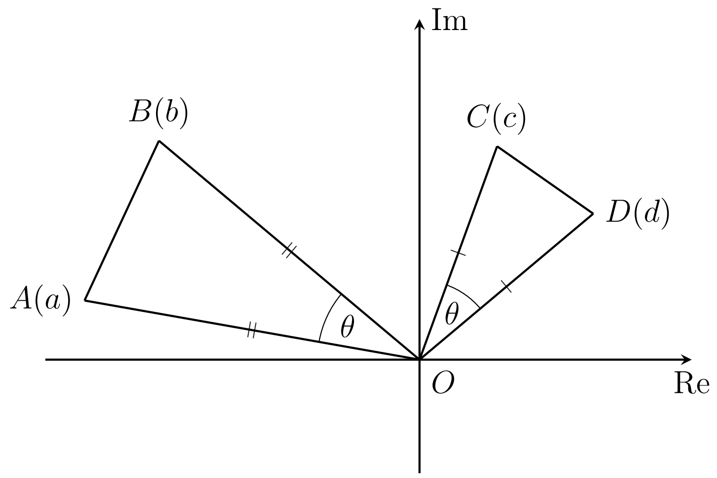
$$\frac{12}{x^3+8} = \frac{A}{x+2} - \frac{Ax - B}{x^2 - 2x + 4}.$$

(ii) Hence, if $a > 1$, show that **3**

$$\int_1^a \frac{12}{x^3+8}\,dx = \frac{1}{2}\ln\left(\frac{a^2+4a+4}{a^2-2a+4}\right) + \sqrt{3}\tan^{-1}\left(\frac{a-1}{\sqrt{3}}\right) - \frac{1}{2}\ln 3.$$

(iii) Find the limiting value of $\displaystyle\int_1^a \frac{12}{x^3+8}\,dx$ as $a \to \infty$. **1**

(b) The points A, B, C and D on the Argand diagram represent the complex numbers a, b, c and d respectively. The triangles ΔOAB and ΔOCD are isosceles with $OA = OB$, $OC = OD$ and $\angle AOB = \angle COD = \theta$, as shown.



(i) Explain why $a = be^{i\theta}$ and $c = de^{i\theta}$. **1**

(ii) Explain, with the aid of a diagram, why the acute angle between the lines AC and BD is equal to $\operatorname{Arg}\left(\frac{c-a}{d-b}\right)$. **2**

(iii) Hence, show that the acute angle between AC and BD is equal to θ. **2**

Question 15 continues on the following page

Question 15 (continued)

(c) For all non-negative real numbers a and b, $(a+b)^2 \geqslant 4ab$. (Do not prove this.)

(i) Using this fact, show that for all non-negative real numbers a, b, and c **2**

$$(a+b+c)^2 \geqslant 8c\sqrt{ab}.$$

(ii) Using part (i), or otherwise, show that for all non-negative real numbers a, b, and c **2**

$$(a+b+c)^3 \geqslant 16\sqrt{2}\,abc.$$

End of Question 15

Question 16 (16 marks) Use a new sheet of paper.

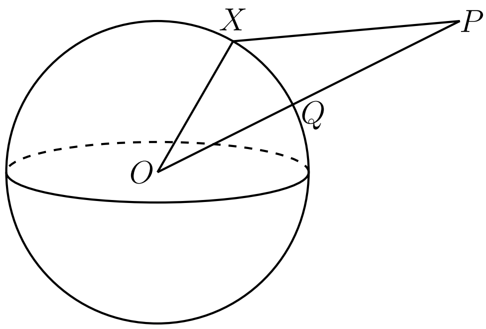
(a) Let a and b be positive integers. **2**

Prove by contradiction that there are no solutions to $a^2 = 9b - 6$.

(b) In the diagram below, P lies outside of a sphere with centre O.

The points Q and X lie on the sphere.

Furthermore, O, Q, and P are collinear.



(i) Use the triangle inequality to show that $XP \geqslant PQ$. **2**

(ii) The real numbers p_1, p_2, p_3, x_1, x_2, and x_3 satisfy **2**

$$p_1{}^2 + p_2{}^2 + p_3{}^2 = 9$$
$$x_1{}^2 + x_2{}^2 + x_3{}^2 = 4.$$

Using part (i), or otherwise, show that

$$(p_1 - x_1)^2 + (p_2 - x_2)^2 + (p_3 - x_3)^2 \geqslant 1.$$

Question 16 continues on the next page

Question 16 (continued)

(c) Let $I_n = \displaystyle\int_0^1 \frac{x^{2n-1}}{(2n-1)!} \sin x \; dx$, where n is an integer, $n \geqslant 1$.

(i) Show that **3**

$$I_n = \frac{\sin 1}{(2n-2)!} - \frac{\cos 1}{(2n-1)!} - I_{n-1}, \quad \text{for } n \geqslant 2.$$

(ii) Using part (i), prove by mathematical induction that for **3**

every positive integer n there exist integers a_n and b_n such that

$$I_n = \frac{a_n \sin 1 + b_n \cos 1}{(2n-1)!}.$$

(iii) It is given that $0 < I_n < \dfrac{1}{(2n)!}$ and $\cos 1 > \dfrac{1}{2}$. (Do not prove this.) **2**

Using these results and part (ii), show that for every positive integer n

there exist integers a_n and b_n such that

$$0 < a_n \tan 1 + b_n < \frac{1}{n}.$$

(iv) Using part (iii), prove by contradiction that $\tan 1$ is irrational. **2**

End of paper

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Question 1	1 mark
The converse of 'If P then Q' is 'If Q then P'. Therefore the converse of 'If $a < b$ then $a^3 < b^3$' is 'If $a^3 < b^3$ then $a < b$'. Hence (D). ✓	

Question 2	1 mark
From the reference sheet (with $f(x) = x$): $$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$$ Therefore: $$\int \frac{1}{\sqrt{2 - x^2}}\,dx = \sin^{-1}\frac{x}{\sqrt{2}} + C$$ Hence (C). ✓	

Question 3	1 mark
$$\begin{aligned} 3a &= \frac{1}{2x^2} \\ a &= \frac{1}{6x^2} \\ \frac{d}{dx}\left(\frac{1}{2}v^2\right) &= \frac{1}{6x^2} \\ \frac{1}{2}v^2 &= -\frac{1}{6x} + C \\ v^2 &= -\frac{1}{3x} + C \end{aligned}$$ Hence (B). ✓	

Question 4	1 mark
Integration by parts:	

$$u = x \qquad v' = e^{3x}$$
$$u' = 1 \qquad v = \frac{1}{3}e^{3x}$$

$$\begin{aligned}\int_0^1 xe^{3x}\,dx &= [uv]_0^1 - \int_0^1 u'v\,dx \\ &= \left[\frac{1}{3}xe^{3x}\right]_0^1 - \frac{1}{3}\int_0^1 e^{3x}\,dx \\ &= \frac{1}{3}e^3 - 0 - \frac{1}{3}\left[\frac{1}{3}e^{3x}\right]_0^1 \\ &= \frac{1}{3}e^3 - \frac{1}{9}\left(e^3 - 1\right) \\ &= \frac{1}{9}(2e^3 + 1)\end{aligned}$$

Hence (D). ✓

Question 5	1 mark

$$\begin{aligned}z &= -\frac{1}{2} - \frac{\sqrt{3}}{2}i \\ &= \text{cis}\left(-\tfrac{2\pi}{3}\right)\end{aligned}$$

$$\begin{aligned}z^9 &= \text{cis}\left(-\tfrac{2\pi}{3}\right)^9 \\ &= \text{cis}\,(-6\pi) \\ &= 1\end{aligned}$$

Hence (A). ✓

Question 6	1 mark
We can rewrite the equation as $\lvert z-(-2)\rvert = \lvert z-(1+i)\rvert$.	

It follows that the solution set is the perpendicular bisector of the interval joining $(-2, 0)$ to $(1, 1)$.

Hence (C). ✓

Question 7	1 mark

$$\begin{aligned}\overrightarrow{EA} + \overrightarrow{AF} + \overrightarrow{FD} &= \overrightarrow{ED}\\ &= \overrightarrow{BC} \qquad \text{(Since } BCDE \text{ is a square.)}\end{aligned}$$

Hence (B). ✓

Question 8	1 mark

By the conjugate root theorem, $1 - 2i$ is also a root.
Let the real root of $P(z)$ be α.
By the sum of the roots:

$$\begin{aligned}\alpha + 1 + 2i + 1 - 2i &= -\frac{b}{a}\\ \alpha + 2 &= -p\\ p &= -\alpha - 2 \qquad \text{(🦕)}\end{aligned}$$

By the product of the roots:

$$\begin{aligned}\alpha(1 + 2i)(1 - 2i) &= -\frac{d}{a}\\ 5\alpha &= -25p\\ \alpha &= -5p\\ \alpha &= -5(-\alpha - 2) \qquad \text{(from (🦕))}\\ 4\alpha &= -10\\ \alpha &= -\frac{5}{2}\end{aligned}$$

Hence (A). ✓

Question 9	1 mark

We want $a + b$ multiplied by something to give $ac - bd$.
This motivates the expansion:

$$(a + b)(c - d) = ac - bd + bc - ad$$

We know that $ac - bd$ is divisible by $a + b$, and we can see that $(a + b)(c - d)$ is divisible by $a + b$.
It follows that $bc - ad$ must also be divisible by $a + b$.

Hence (C). ✓

Question 10	1 mark
Notice that $\underset{\sim}{a}\cdot\underset{\sim}{b}$ and $\lvert\underset{\sim}{a}\rvert$ are (non-zero) scalars. Therefore, $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are scalar multiples of each other. This means they are either pointing in the same direction ($\theta = 0$) or opposite directions ($\theta = \pi$). $$\begin{aligned}(\underset{\sim}{a}\cdot\underset{\sim}{b})\,\underset{\sim}{a} &= \lvert\underset{\sim}{a}\rvert\,\underset{\sim}{b}\\ \cancel{\lvert\underset{\sim}{a}\rvert}\lvert\underset{\sim}{b}\rvert\cos\theta\,\underset{\sim}{a} &= \cancel{\lvert\underset{\sim}{a}\rvert}\,\underset{\sim}{b}\\ \pm\underset{\sim}{a} &= \frac{\underset{\sim}{b}}{\lvert\underset{\sim}{b}\rvert} \qquad (\cos\theta = \pm 1,\text{ since }\theta = 0,\pi)\\ \underset{\sim}{a} &= \pm\hat{\underset{\sim}{b}}\end{aligned}$$ Hence (B). ✓	

Question 11

(a)

$$\frac{\bar{u}}{iv} = \frac{\overline{5i}}{i(-5-i)}$$

$$= \frac{-5i}{1-5i} \times \frac{1+5i}{1+5i}$$

$$= \frac{25-5i}{1+25}$$

$$= \frac{25}{26} - \frac{5}{26}i$$

2 marks

1st mark for showing step to realise the denominator

Comment:

Well done though some students didn't convert to Cartesian form.

(b) (i)

$$\text{Let } a + ib = \sqrt{7+24i}$$

$$a^2 - b^2 + 2abi = 7 + 24i$$

$$a^2 - b^2 = 7 \quad ; \quad ab = 12$$

By inspection, $a = \pm 4\,, b = \pm 3$

$$\therefore \quad \sqrt{7+24i} = \pm(4+3i)$$

2 marks

1st mark for a reasonable attempt with minor errors

Comment:

Well done though some students made errors with a and b having opposite signs.

(ii) $2z^2 - \sqrt{7}\,z - 3i = 0$

$$z = \frac{\sqrt{7} \pm \sqrt{\left(-\sqrt{7}\,\right)^2 - 4(2)(-3i)}}{2(2)}$$

$$= \frac{\sqrt{7} \pm \sqrt{7+24i}}{4}$$

$$\therefore \quad z = \frac{\sqrt{7} \pm (4+3i)}{4}$$

2 marks

1st mark for showing correct application of the quadratic formula

Comment:

Well done though some students made errors when applying the quadratic formula. Some students could have provided better working to clearly show the process.

(c)

$$\int \frac{1}{1+e^{-x}}\,dx = \int \frac{e^x}{e^x+1}\,dx$$
$$= \ln(e^x+1) + c$$

<u>2 marks</u>

1st mark for multiplying by $\frac{e^x}{e^x}$ or manipulating the fraction in an appropriate form for integration.

<u>Comment:</u>

Well done though some students used less efficient methods, including the unnecessary use of partial fractions.

(d)

Let $x = \tan\theta$, $dx = \sec^2\theta\ d\theta$.

$$\int \frac{1}{(1+x^2)^{\frac{3}{2}}}\,dx = \int \frac{1}{(1+\tan^2\theta)^{\frac{3}{2}}}\sec^2\theta\ d\theta$$
$$= \int \frac{1}{(\sec^2\theta)^{\frac{3}{2}}}\sec^2\theta\ d\theta$$
$$= \int \frac{1}{(\sec^2\theta)^{\frac{1}{2}}}\ d\theta$$
$$= \int \cos\theta\ d\theta$$
$$= \sin\theta + c$$
$$= \frac{x}{\sqrt{1+x^2}} + c$$

<u>3 marks</u>

1st mark for setting up the correct substitution.

2nd marks for tidying up the expression in the integral to $\cos\theta$.

Final mark for converting back to the variable x.

<u>Comment:</u>

Well done by some students but there were a number of students who did not recognise the appropriate substitution needed for this type of integral. It is as standard form for using $x = \tan\theta$ and students who did not recognise this are advised to review this area of integration. This was the worst attempted part of Q11.

(e) If two lines are perpendicular, their dot product must be zero.

$$1 \times -4 + a \times 4 + b \times 6 = 0$$

$$2a + 3b = 2 \quad ❶$$

$$1 \times 7 + a \times 3 + b \times 2 = 0$$

$$3a + 2b = -7 \quad ❷$$

Solving simultaneously $3 \times$ ❶ $- 2 \times$ ❷ :

$$9b - 4b = 3 \times 2 - 2 \times (-7)$$

$$5b = 20$$

$$b = 4$$

$$2a + 3(4) = 2$$

$$2a = -10$$

$$\therefore \quad a = -5 \;, \quad b = 4$$

<u>2 marks</u>

1st mark for correctly setting up the two equations needed.

<u>Comment:</u>

Well done though some students made some unfortunate arithmetic errors.

(f)

$$\cos 60° = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$

$$= \frac{p \times p + 2 \times 0 + 0 \times 2}{\sqrt{p^2 + 4} \times \sqrt{p^2 + 4}}$$

$$\frac{1}{2} = \frac{p^2}{p^2 + 4}$$

$$p^2 + 4 = 2p^2$$

$$p^2 = 4$$

$$\therefore \quad p = \pm 2$$

<u>2 marks</u>

1st mark for correct application of the relevant formula.

<u>Comment:</u>

Well done though some students incorrectly omitted the $\pm$ for the values of p.

Question 12(a)

· Provides correct solution.	3 marks
· Obtains equation (🦎) and evaluates one integral correctly.	2 marks
· Completes the square in the denominator.	1 mark

Notice that $\frac{d}{dx}\sqrt{4-2x-x^2} = \dfrac{-1-x}{\sqrt{4-2x-x^2}}$.

This motivates the following algebraic manipulation:

$$\int_0^1 \frac{x+3}{\sqrt{4-2x-x^2}}\,dx$$

$$= \int_0^1 \frac{2}{\sqrt{5-(x+1)^2}}\,dx - \int_0^1 \frac{-1-x}{\sqrt{4-2x-x^2}}\,dx \quad (🦎)$$

$$= \left[2\sin^{-1}\frac{x+1}{\sqrt{5}}\right]_0^1 - \left[\sqrt{4-2x-x^2}\right]_0^1$$

$$= 2\sin^{-1}\frac{2}{\sqrt{5}} - 2\sin^{-1}\frac{1}{\sqrt{5}} + 1$$

Generally well done. There was some difficulty with completing the square and finding the integral of $\frac{-1-x}{\sqrt{4-2x-x^2}}$.

Question 12(b)

· Provides correct solution.	3 marks
· Writes $\cos^5 x = (1-\sin^2 x)^2 \cos x$ and expands correctly.	2 marks
· Applies the Pythagorean identity $\sin^2 x + \cos^2 x = 1$.	1 mark

$$\int \sin^4 x \cos^5 x\,dx$$

$$= \int \sin^4 x(1-\sin^2 x)^2 \cos x\,dx$$

$$= \int \sin^4 x(1-2\sin^2 x + \sin^4 x)\cos x\,dx$$

$$= \int \left(\sin^4 x\cos x - 2\sin^6 x\cos x + \sin^8 x\cos x\right)\,dx$$

$$= \frac{\sin^5 x}{5} - \frac{2\sin^7 x}{7} + \frac{\sin^9 x}{9} + C$$

Applying the Pythagorean identity $\sin^2 x + \cos^2 x = 1$ was required to make progress. The substitution $u = \sin x$ was used fairly successfully.

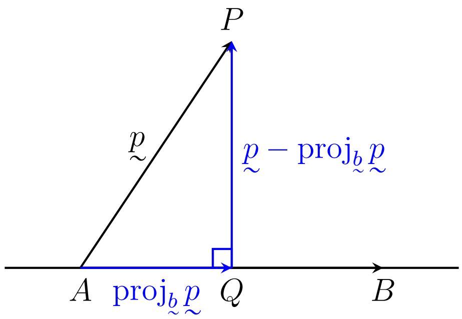
Question 12(c)(i)

· Finds $\overrightarrow{AP}$ and $\overrightarrow{AB}$ correctly.	1 mark
$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$ $= \begin{bmatrix} 0 \\ -1 \\ -4 \end{bmatrix} - \begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix}$ $= \begin{bmatrix} 1 \\ 1 \\ -8 \end{bmatrix}$ $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ $= \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix}$ $= \begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix}$ ✓	Well done. Some issues with notation of vectors, e.g. students writing vectors as $(1, 1, -8)$ instead of $\begin{bmatrix} 1 \\ 1 \\ -8 \end{bmatrix}$.

Question 12(c)(ii)

· Finds $\text{proj}_{\underset{\sim}{b}} \underset{\sim}{p}$ correctly.	2 marks
· States a correct formula for $\text{proj}_{\underset{\sim}{b}} \underset{\sim}{p}$.	1 mark
$\text{proj}_{\underset{\sim}{b}} \underset{\sim}{p} = \dfrac{\underset{\sim}{p} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{b}} \underset{\sim}{b}$ ✓ $= \dfrac{\begin{bmatrix} 1 \\ 1 \\ -8 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix}}{\begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix}} \begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix}$ $= \dfrac{4 + 7 + 8}{16 + 49 + 1} \begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix}$ $= \dfrac{19}{66} \begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix}$ ✓	A common error was forgetting to square the square root arising from the denominator. Carried errors from (c)(i) were not penalised.

Question 12(c)(iii)

· Finds the distance from P to AB correctly.	2 marks
· States that the required distance is $\lvert \underset{\sim}{p} - \text{proj}_{\underset{\sim}{b}}\, \underset{\sim}{p} \rvert$.	1 mark
 As seen in the diagram above, the required distance is $\begin{aligned} \left\lvert \underset{\sim}{p} - \text{proj}_{\underset{\sim}{b}}\, \underset{\sim}{p} \right\rvert &= \left\lvert \begin{bmatrix} 1 \\ 1 \\ -8 \end{bmatrix} - \frac{19}{66} \begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix} \right\rvert \\ &= \frac{1}{66} \left\lvert \begin{bmatrix} -10 \\ -67 \\ -509 \end{bmatrix} \right\rvert \\ &= \frac{1}{66}\sqrt{10^2 + 67^2 + 509^2} \\ &\approx 7.7801 \ (4\,\text{d.p.}) \end{aligned}$	A clear diagram was important to understand this question. Some students mistakenly found $\lvert \text{proj}_{\underset{\sim}{b}}\, \underset{\sim}{p} \rvert$ or $\left\lvert \underset{\sim}{b} - \text{proj}_{\underset{\sim}{b}}\, \underset{\sim}{p} \right\rvert$

Question 12(d)(i)

· Provides correct proof.	2 marks
· Applies Euler's formula correctly.	1 mark
$\text{RHS} = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$ $= \frac{1}{2i}\big(\cos\theta + i\sin\theta - (\cos(-\theta) + i\sin(-\theta))\big)$ ✓ $= \frac{1}{2i}\left(\cos\theta + i\sin\theta - (\cos\theta - i\sin\theta)\right)$ $= \frac{1}{2i}(2i\sin\theta)$ $= \sin\theta$ ✓ $= \text{LHS}$	Very well done. Students showed the result carefully using that cosine is even and sine is odd.

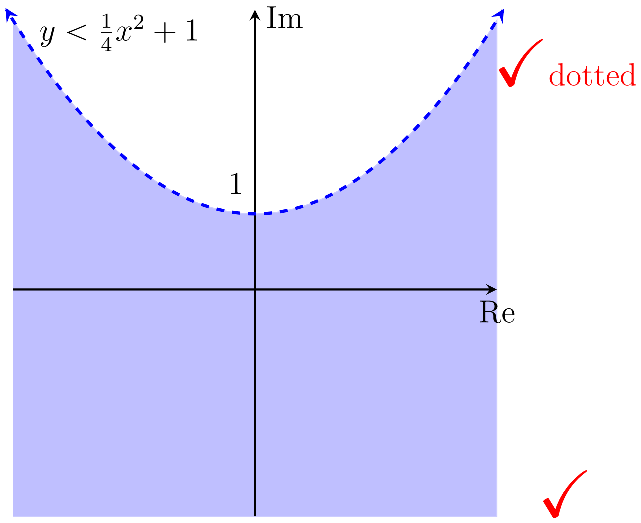
Question 12(d)(ii)

· Provides correct proof.	2 marks
· Expands the perfect cube correctly.	1 mark
$\text{LHS} = \sin^3\theta$ $= \left(\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\right)^3$ $= -\frac{1}{8i}\left(e^{3i\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-3i\theta}\right)$ ✓ $= \frac{1}{8i}\left(3e^{i\theta} - 3e^{-i\theta} - e^{3i\theta} + e^{-3i\theta}\right)$ $= \frac{1}{4}\left[\frac{3}{2i}\left(e^{i\theta} - e^{-i\theta}\right) - \frac{1}{2i}\left(e^{3i\theta} - e^{-3i\theta}\right)\right]$ $= \frac{1}{4}(3\sin\theta - \sin 3\theta)$ ✓ $= \text{RHS}$	Students struggled to expand the perfect cube. The preferred method is simply to find the appropriate row of Pascal's triangle (1, 3, 3, 1) and apply the binomial theorem. Students who expanded 'by hand' often ran into critical algebraic errors, which shows the importance of revising the binomial theorem.

Question 13(a)

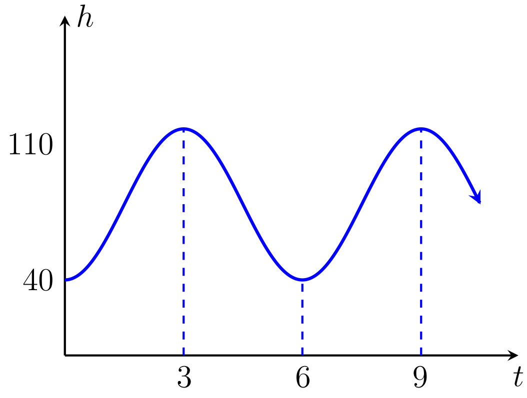
· Provides correct solution.	3 marks
· Simplifies the resulting integrand.	2 marks
· Applies the method of t-substitution.	1 mark
Let $t = \tan\frac{x}{2}$ $dt = \frac{1}{2}\sec^2\frac{x}{2}\ dx$ $dt = \frac{1}{2}(1+\tan^2\frac{x}{2})\ dx$ $dx = \frac{2}{1+t^2}\,dt$ ✓ When $x = \frac{\pi}{3}$, $t = \frac{1}{\sqrt{3}}$ When $x = \frac{\pi}{2}$, $t = 1$ $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{1-\cos x}\,dx = \int_{\frac{1}{\sqrt{3}}}^{1} \frac{1}{1-\frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\,dt$ $= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{2}{1+t^2-(1-t^2)}\,dt$ ✓ $= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{2}{2t^2}\,dt$ $= \left[-\frac{1}{t}\right]_{\frac{1}{\sqrt{3}}}^{1}$ $= \sqrt{3}-1$ ✓	This part was quite challenging, with many students not knowing which technique to apply. Some students found elegant alternative approaches based on the double angle identity $1-\cos x = 2\sin^2\frac{x}{2}$, or by multiplying top and bottom by $1+\cos x$ and splitting the fraction.

Question 13(b)

· Provides correct sketch of region.	3 marks
· Obtains correct dotted sketch of parabola.	2 marks
· Substitutes $z = x + iy$ and applies formula for the modulus.	1 mark
Let $z = x + iy$: $$\|z - 2i\| > \operatorname{Im} z$$ $$\sqrt{x^2 + (y-2)^2} > y$$ $$x^2 + (y-2)^2 > y^2$$ $$x^2 - 4y + 4 > 0$$ $$y < \frac{1}{4}x^2 + 1$$ 	This part was very challenging, with great attention to detail required for full marks. Common mistakes included forgetting to square both sides of the inequality when removing the square root, forgetting to give a dotted sketch of the parabola, and incorrectly or incompletely shading the required region.

Question 13(c)(i)

· Provides correct solution.	2 marks
· Finds exactly one of c, a and n.	1 mark



By observing the sketch, we see that:

$$c = \frac{110 + 40}{2}$$

$$= 75$$

Also, because the curve is an inverted cosine shape, a is **negative**.

$$\text{amplitude} = \frac{110 - 40}{2}$$

$$= 35$$

Therefore $a = -35$.
Finally we see the period is 6 seconds, which implies:

$$6 = \frac{2\pi}{n}$$

$$6n = 2\pi$$

$$n = \frac{\pi}{3}$$

Therefore $h = 75 - 35\cos\frac{\pi t}{3}$.

Many students were misled by the unusual labelling of the equation into thinking that $a = 35$. This was not penalised in cases where students made it clear that they understood the shape of the curve is an inverted cosine shape (through a diagram, or through working out).

Question 13(c)(ii)

· Provides correct solution.	2 marks
· Obtains $\cos\frac{\pi t}{3} < -\frac{5}{7}$.	1 mark
We require that $h > 100$ during the first six seconds: $$\begin{aligned} h &> 100 \\ 75 - 35\cos\frac{\pi t}{3} &> 100 \\ -35\cos\frac{\pi t}{3} &> 25 \\ \cos\frac{\pi t}{3} &< -\frac{5}{7} \checkmark \end{aligned}$$ Because cosine is negative quadrants II and III: $$\begin{aligned} \pi - \cos^{-1}\frac{5}{7} < \frac{\pi t}{3} < \pi + \cos^{-1}\frac{5}{7} \\ \frac{3}{\pi}(\pi - \cos^{-1}\frac{5}{7}) < t < \frac{3}{\pi}(\pi + \cos^{-1}\frac{5}{7}) \\ 2.25974\ldots < t < 3.740255\ldots \end{aligned}$$ Therefore the required period of time is $3.740255... - 2.25974... \approx 1.48$ seconds. (2 d.p.) $\checkmark$	Due to the large number of carried errors into this part, it was marked fairly leniently. The first mark was given for attempting to solve $\cos\frac{\pi t}{3} = -\frac{5}{7}$, and the second mark was given for showing understanding that the required time interval lies between the two solutions to the equation.

Question 13(d)(i)

· Provides correct solution.	2 marks
· Obtains a correct expression for one of $\overrightarrow{AM}$ or $\overrightarrow{AN}$.	1 mark
From the diagram we have: $$\begin{aligned} \overrightarrow{AM} &= \frac{1}{2}\overrightarrow{AC} \\ &= \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{BC}) \\ &= \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b}) \checkmark \end{aligned}$$ $$\begin{aligned} \overrightarrow{AN} &= \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BD} \\ &= \underset{\sim}{a} + \frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{c}) \checkmark \end{aligned}$$	There were many successful approaches to this question. Superior responses used the fewest number of steps to find $\overrightarrow{AM}$. Inferior responses took circuitous routes to go from A to M, which needlessly complicated things, and occasionally lead to avoidable algebraic errors.

Question 13(d)(ii)

· Provides correct solution.	2 marks
· Shows that $\underset{\sim}{a} + \underset{\sim}{c} = 2\underset{\sim}{b}$ or $\overrightarrow{MN} = \frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{c}$.	1 mark
Observe that: $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD}$ $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} = 3\underset{\sim}{b}$ $\underset{\sim}{a} + \underset{\sim}{c} = 2\underset{\sim}{b}$ $\checkmark$ (🐉) Therefore: $\overrightarrow{MN} = \overrightarrow{AN} - \overrightarrow{AM}$ $= \underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{c} - \left(\frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}\right)$ $= \frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{c}$ $= \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{c})$ $= \frac{1}{2}(2\underset{\sim}{b})$ (in view of (🐉)) $= \underset{\sim}{b}$ $\checkmark$	Both parts of this question were quite well done. Some students found a beautiful solution which completely avoided the use of the vector $\underset{\sim}{c}$. The idea was to first write $\overrightarrow{AM} = \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})$ and $\overrightarrow{AN} = \frac{1}{2}(\underset{\sim}{a} + 3\underset{\sim}{b})$ so that $\overrightarrow{MN} = \frac{1}{2}(\underset{\sim}{a}+\underset{\sim}{b}) - \frac{1}{2}(\underset{\sim}{a}+3\underset{\sim}{b}) = \underset{\sim}{b}$.

$\boxed{Q14}$

a) $x^2 + y^2 + z^2 = 10$

$$\left| \underset{\sim}{r} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right| = 10$$

$$\underset{\sim}{r} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\left| \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right| = 10$$

$$\left| \begin{bmatrix} 1 + \lambda \times 0 \\ 2 + \lambda \\ 1 - \lambda \end{bmatrix} \right| = 10$$

$$1^2 + (2+\lambda)^2 + (1-\lambda)^2 = 10$$

$$1 + \lambda^2 + 4\lambda + 4 + 1 - 2\lambda + \lambda^2 = 10$$

$$2\lambda^2 + 2\lambda - 4 = 0$$

$$2(\lambda^2 + \lambda - 2) = 0$$

$$(\lambda + 2)(\lambda - 1) = 0$$

$$\lambda = -2 \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\lambda = 1 \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

The points of intersection
$A(1, 0, 3)$ and $B(1, 3, 0)$

$\boxed{Q14}$ a/ii) $\vec{OA} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\vec{OB} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$

let $\angle AOB = \theta$

$|\vec{OA}| = \sqrt{1^2 + 0^2 + 3^2} = \sqrt{10}$

$|\vec{OB}| = \sqrt{1^2 + 3^2 + 0^2} = \sqrt{10}$

$\vec{OA} . \vec{OB} = 1 + 0 + 0 = 1$

$\vec{OA} . \vec{OB} = |\vec{OA}| . |\vec{OB}| . \cos\theta$

$\cos\theta = \dfrac{\vec{OA} . \vec{OB}}{|\vec{OA}| |\vec{OB}|} = \dfrac{1}{\sqrt{10} . \sqrt{10}} = \dfrac{1}{10}$

$\theta = 84^\circ$ OR $\angle AOB = 84^\circ$

b) i) $\ddot{x} = -9(x-7)$

$\dfrac{d(\frac{1}{2}v^2)}{dx} = -9(x-7)$

$\int d(\frac{1}{2}v^2) = -9\int(x-7)\,dx$

$\dfrac{v^2}{2} = -9\left(\dfrac{x^2}{2} - 7x\right) + D$

$v^2 = -18\left(\dfrac{x^2 - 14x}{2}\right) + D$

$t=0,\ x=0,\ v = 28 \therefore D = 28^2 = 784$

$v^2 = -9(x^2 - 14x) + 784$

$\boxed{Q14}$ b/ii) $v^2 = -9(x^2 - 14x) + 784$

$$v^2 = -9[(x-7)^2 - 49] + 784$$

$$v^2 = -9(x-7)^2 + 1225$$

$$v^2 = n^2[a^2 - (x-c)^2]$$

$$v^2 = 9\left[\frac{1225}{9} - (x-7)^2\right]$$

$$v^2 = 9\left[\left(\frac{35}{3}\right)^2 - (x-7)^2\right]$$

Max Velocity: $x - c = 0$

$x - 7 = 0 \quad \therefore x = 7$

$$v^2 = 9\left[\left(\frac{35}{3}\right)^2 - 0\right]$$

$$v = 35 \text{ m/s}$$

c(i) let $z = \cos\theta + i\sin\theta$

$$z^5 = (\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$$

$$(\cos\theta + i\sin\theta)^5 = {}^5C_0\cos\theta + {}^5C_1\cos^4\theta\, i\sin\theta + {}^5C_2\cos^3\theta.(i\sin\theta)^2 + {}^5C_3\cos^2\theta.(i\sin\theta)^3 + {}^5C_4\cos\theta.(i\sin\theta)^4 + {}^5C_5(i\sin\theta)^5$$

$$= \cos\theta + 5\cos^4\theta\, i\sin\theta + 10\cos^3\theta\sin^2\theta + 5\cos\theta.\sin^4\theta - 10\cos^2\theta.\sin^3\theta\, i + i\sin^5\theta$$

Equating the imaginary part:

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta.\sin^3\theta + \sin^5\theta$$

Q14

c(ii) $\sin 5\theta = 5\sin\theta\left[(1-\sin^2\theta)^2\right] - 10\sin^3\theta(1-\sin^2\theta) + \sin^5\theta$

$\sin 5\theta = 5\sin\theta(1-2\sin^2\theta+\sin^4\theta) - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$

$\sin 5\theta = 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$

$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ (1)

c(iii) let $x = \sin\theta$:

$16x^5 - 20x^3 + 5x = 1$

OR $16\sin^5\theta - 20\sin^3\theta + 5\sin\theta = 1 = \sin 5\theta$ from (1)

$\sin 5\theta = 1$

$5\theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2} \ldots,$

$\theta = \frac{\pi}{10}, \frac{\pi}{2}, \frac{9\pi}{10}, \frac{13\pi}{10}$

Thus $x = \sin\frac{\pi}{10}, \sin\frac{\pi}{2}, \sin\frac{13\pi}{10}$

$x = \sin\frac{\pi}{10}, 1, \sin\left(\pi + \frac{3\pi}{10}\right)$

The roots are: $\sin\frac{\pi}{10}, 1, \sin\left(-\frac{3\pi}{10}\right)$

c(iv) Since $x = \sin\frac{\pi}{10}$ and $x = \sin\left(-\frac{3\pi}{10}\right) = -\sin\frac{3\pi}{10}$ are double root.

The product of the roots:

$\sin\frac{\pi}{10}\cdot\sin\frac{\pi}{10}\cdot\left(-\sin\frac{3\pi}{10}\right)\cdot\left(-\sin\frac{3\pi}{10}\right) = \frac{-d}{a} = \frac{1}{16}$

$\sin^2\frac{\pi}{10}\cdot\sin^2\frac{3\pi}{10} = \frac{1}{16}$

OR $\sin\frac{\pi}{10}\cdot\sin\frac{3\pi}{10} = \frac{1}{4}$

Question 15

(a) (i) (2 marks)

✓ [1] for correct value of A

✓ [1] for correct value of B

$$12 = A(x^2 - 2x + 4) - (Ax - B)(x + 2)$$

Let $x = -2$

$$12 = 12A \qquad \therefore A = 1$$

Let $x = 0$

$$12 = 4A + B(2)$$

$$8 = 2B \qquad \therefore B = 4$$

(ii) (3 marks)

✓ [1] for obtaining $\displaystyle\int_1^a \frac{1}{x+2} - \frac{1}{2}\frac{2x-2}{x^2-2x+4} + \frac{3}{x^2-2x+4}\,dx$, or equivalent merit

✓ [1] for obtaining $\displaystyle\left[\ln(x+2) - \frac{1}{2}\ln(x^2-2x+4) + \frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right)\right]_1^a$

✓ [1] for correctly showing the required result

$$\begin{aligned}
&\int_1^a \frac{12}{x^3+8}\,dx \\
&= \int_1^a \frac{1}{x+2} - \frac{x-4}{x^2-2x+4}\,dx \\
&= \int_1^a \frac{1}{x+2} - \frac{1}{2}\frac{2x-2}{x^2-2x+4} + \frac{3}{x^2-2x+4}\,dx \\
&= \int_1^a \frac{1}{x+2} - \frac{1}{2}\frac{2x-2}{x^2-2x+4} + \frac{3}{(x-1)^2+3}\,dx \\
&= \left[\ln(x+2) - \frac{1}{2}\ln(x^2-2x+4) + \frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right)\right]_1^a \\
&= \ln(a+2) - \frac{1}{2}\ln(a^2-2a+4) + \frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{a-1}{\sqrt{3}}\right) - \left(\ln 3 - \frac{1}{2}\ln 3 + \frac{3}{\sqrt{3}}\tan^{-1} 0\right) \\
&= \frac{1}{2}\big(2\ln(a+2) - \ln(a^2-2a+4)\big) + \sqrt{3}\tan^{-1}\left(\frac{a-1}{\sqrt{3}}\right) - \frac{1}{2}\ln 3 \\
&= \frac{1}{2}\big(\ln(a+2)^2 - \ln(a^2-2a+4)\big) + \sqrt{3}\tan^{-1}\left(\frac{a-1}{\sqrt{3}}\right) - \frac{1}{2}\ln 3 \\
&= \frac{1}{2}\ln\left(\frac{a^2+4a+4}{a^2-2a+4}\right) + \sqrt{3}\tan^{-1}\left(\frac{a-1}{\sqrt{3}}\right) - \frac{1}{2}\ln 3
\end{aligned}$$

Comment:

- Some students had trouble obtaining the second and third integrand due to difficulties with algebra. When manipulating integrands, it is advised to verify whether the current line of working is equal to the previous line of working.

(iii) (1 mark)

✓ [1] for correct final answer

$$\begin{aligned}
&\text{As } a \longrightarrow \infty, \\
&\ln\left(\frac{a^2+4a+4}{a^2-2a+4}\right) \longrightarrow \ln 1 = 0 \\
&\tan^{-1}\left(\frac{a-1}{\sqrt{3}}\right) \longrightarrow \frac{\pi}{2} \\
&\therefore \int_1^a \frac{12}{x^3+8}\,dx \longrightarrow \frac{\sqrt{3}\pi}{2} - \frac{1}{2}\ln 3
\end{aligned}$$

Comment:

- Some students were not able to deduce that $\dfrac{a^2+4a+4}{a^2-2a+4} \longrightarrow 1$, as $a \longrightarrow \infty$.
- Some students did not know that $\tan^{-1}(x) \longrightarrow \dfrac{\pi}{2}$ as $x \longrightarrow \infty$.

(b) (i) (1 mark)

✓ [1] for correct explanation of why $a = be^{i\theta}$ OR why $c = de^{i\theta}$

$\overrightarrow{OA}$ is $\overrightarrow{OB}$ rotated θ anticlockwise

$$\therefore \overrightarrow{OA} = \overrightarrow{OB} \times e^{i\theta}$$
$$\therefore a = be^{i\theta}$$

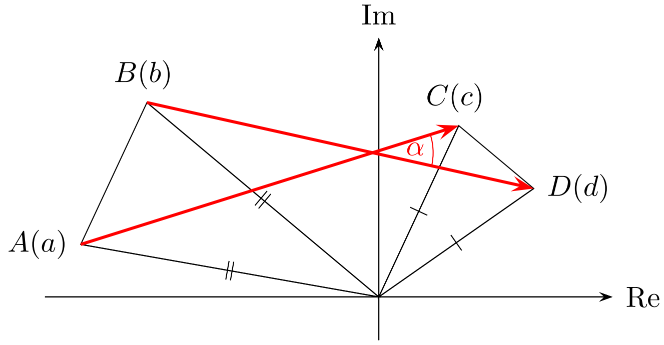
Similarly,

$$\therefore \overrightarrow{OC} = \overrightarrow{OD} \times e^{i\theta}$$
$$\therefore c = de^{i\theta}$$

(ii) (2 marks)

✓ [1] for diagram with the inclusion of lines AC and BD

✓ [1] for correct explanation



$$\begin{aligned}\alpha &= \arg \overrightarrow{AC} - \arg \overrightarrow{BD} \\ &= \arg(c-a) - \arg(d-b) \\ &= \arg\left(\frac{c-a}{d-b}\right)\end{aligned}$$

Comment:

- Better responses included a labelled diagram, and have started with the fact that α is the difference between $\arg(\overrightarrow{AC})$ and $\arg(\overrightarrow{BD})$.

(iii) (2 marks)

✓ [1] for using part (i) in part (ii)

✓ [1] for correct proof

$$\begin{aligned}\alpha &= \arg\left(\frac{c-a}{d-b}\right) \\ &= \arg\left(\frac{de^{i\theta} - be^{i\theta}}{d-b}\right) \\ &= \arg\left(\frac{e^{i\theta}(d-b)}{d-b}\right) \\ &= \arg(e^{i\theta}) \\ &= \theta\end{aligned}$$

(c) (i) (2 marks)

✓ [1] for using $(a+b)^2 \geq 4ab$ to obtain $(a+b+c)^2 \geq 4(a+b)c$

✓ [1] for deducing the required result

$$\begin{aligned}(a+b+c)^2 &= \big((a+b)+c\big)^2 \\ &\geq 4(a+b)c \\ &\geq 4 \times 2\sqrt{abc} \\ &= 8c\sqrt{ab} \qquad (1)\end{aligned}$$

(ii) (2 marks)

✓ [1] for writing down the two similar results

✓ [1] for deducing the required result

Similarly,

$$\begin{aligned}(a+b+c)^2 &\geq 8a\sqrt{bc} \qquad (2) \\ (a+b+c)^2 &\geq 8b\sqrt{ac} \qquad (3)\end{aligned}$$

$(1) \times (2) \times (3)$:

$$\begin{aligned}(a+b+c)^6 &\geq 8^3 abc\sqrt{a^2b^2c^2} \\ &= 8^3a^2b^2c^2 \\ \therefore (a+b+c)^3 &\geq 8\sqrt{8}abc \quad \text{(on taking the square root of both sides)} \\ &= 16\sqrt{2}abc\end{aligned}$$

Alternatively,

$$\begin{aligned}\frac{a+b+c}{3} &\geq \sqrt[3]{abc} \\ (a+b+c)^3 &\geq 27abc \\ &\geq 16\sqrt{2}abc\end{aligned}$$

Comment:

- Only a handful of students were able to receive full marks for this part. Better responses included writing down the two similar results. This is a common technique that students should remember as part of their problem solving techniques.

- A small number of students also proved the result using the AM-GM inequality for 3 terms. This result may be freely stated and used without proof (unless otherwise specified – e.g. a question is asking for the proof of the AM-GM inequality). Students are advised to remember and use, where appropriate, the extended AM-GM inequality.

Question 16

(a) Assume that there is at least one solution to

$$a^2 = 9b - 6$$

where a and b are both positive integers.

$$a^2 = 3(3b - 2)$$

Since a is a positive integer, a^2 must have prime factors in pairs. Hence $(3b-2)$ must have 3 as a factor so that a^2 has 3^2 as a factor.

Since b is a positive integer, $(3b-2)$ is two less than a multiple of 3 and hence, $(3b-2)$ is NOT a multiple of 3.

Hence, 3 is a factor of a^2 but 3^2 is not and this contradicts the assumption that a is a positive integer.

∴ By contradiction, there is no solution to

$$a^2 = 9b - 6$$

where a and b are both positive integers.

<u>2 marks</u>

1st mark for both:

- correct assumption
- factorisation of $9b - 6$ in some form.

<u>Comment:</u>

Not well attempted by the majority of students. Many students attempted to work with cases of odd and even integers which was not an effective approach for this question due to its relationship to multiples of 3 (not multiples of 2).

(b) (i) By the triangle inequality:

$$OX + XP \geq OP \text{ (using } \Delta OXP)$$

$$\text{i.e. } OX + XP \geq OQ + QP$$

$$r + XP \geq r + QP \text{ (Since } OX = OQ = r)$$

$$\therefore \; XP \geq PQ$$

<u>2 marks</u>

1 mark for either:

- appropriate version of triangle inequality
- recognition of common radii

<u>Comment:</u>

Many students were effective in proving the required relationship though the structure of working can be improved to provide greater clarity for the reader.

(b) (ii) Let the coordinates of the points O, P and X be:

$$O = (0,0,0), \;\; X = (x_1, x_2, x_3) \;\text{ and }\; P(p_1, p_2, p_3)$$

Let the radius of the sphere in the diagram be 2 units.

Hence, $OX = 2$.

$$(x_1 - 0)^2 + (x_2 - 0)^2 + (x_3 - 0)^2 = 2^2.$$
$$\text{i.e. } {x_1}^2 + {x_2}^2 + {x_3}^2 = 4.$$

Now consider OP.

$$OP^2 = (p_1 - 0)^2 + (p_2 - 0)^2 + (p_3 - 0)^2$$
$$\text{i.e. } OP^2 = {p_1}^2 + {p_2}^2 + {p_3}^2$$

Since P lies outside the circle, $OP > 2$.

Choose P such that $OP = 3$ and $OP^2 = 3^2$.

Hence:

$${p_1}^2 + {p_2}^2 + {p_3}^2 = 9$$

Consider $XP \geq PQ$ (as proven in part (i))

$$XP^2 \geq PQ^2$$
$$(p_1 - x_1)^2 + (p_2 - x_2)^2 + (p_3 - x_3)^2 \geq (OP - OQ)^2$$
$$(p_1 - x_1)^2 + (p_2 - x_2)^2 + (p_3 - x_3)^2 \geq (3 - 2)^2$$

$\therefore \;\; (p_1 - x_1)^2 + (p_2 - x_2)^2 + (p_3 - x_3)^2 \geq 1$

<u>2 marks</u>

1 mark for either:

- A good attempt at linking the elements of the question to the sphere.
- A good attempt at using the result in part (i) $XP > PQ$ where XP is the distance between two points X and P.

<u>Comment:</u>

Many students produced confused working that indicated a lack of understanding about the points in the diagram and the relationships provided in the question.

Some students attempted to use distance between points with no reference to the square root sign and were not awarded full marks.

An alternative and infrequently used approach involved an expansion of LHS of the inequality to be proven, along with use of the AM/GM inequality.

(c) (i) Let $u = \dfrac{x^{2n-1}}{(2n-1)!}$ $\qquad v' = \sin x$

$$u' = \frac{(2n-1)x^{2n-2}}{(2n-1)!} \qquad v = -\cos x$$

$$= \frac{(2n-1)x^{2n-2}}{(2n-1)\times(2n-2)!}$$

$$= \frac{x^{2n-2}}{(2n-2)!}$$

$$I_n = \left[\frac{x^{2n-1}}{(2n-1)!} \times -\cos x\right]_0^1 - \int_0^1 \frac{x^{2n-2}}{(2n-2)!} \times -\cos x\,dx$$

$$= -\frac{\cos 1}{(2n-1)!} - (-0) + \int_0^1 \frac{x^{2n-2}}{(2n-2)!} \times \cos x\,dx$$

Let $u = \dfrac{x^{2n-2}}{(2n-2)!}$ $\qquad v' = \cos x$

$$u' = \frac{(2n-2)x^{2n-3}}{(2n-2)!} \qquad v = \sin x$$

$$= \frac{(2n-2)x^{2n-3}}{(2n-2)\times(2n-3)!}$$

$$= \frac{x^{2n-3}}{(2n-3)!}$$

$$\int_0^1 \frac{x^{2n-2}}{(2n-2)!} \times \cos x\,dx$$

$$= \left[\frac{x^{2n-2}}{(2n-2)!} \times \sin x\right]_0^1 - \int_0^1 \frac{x^{2n-3}}{(2n-3)!} \times \sin x\,dx$$

$$= \frac{\sin 1}{(2n-2)!} - 0 - \int_0^1 \frac{x^{2(n-1)-1}}{(2(n-1)-1)!} \times \sin x\,dx$$

$$= \frac{\sin 1}{(2n-2)!} - I_{n-1}$$

<u>3 marks</u>

- 1st mark for correctly setting up the integration by parts needed
- 2nd mark for recognition of 2nd integration by parts

Comment:
Well done in general. The best solutions provided a very clear link to I_{n-1} by writing $2n-3$ in the form $2(n-1)-1$.
Some students setup the 1st IBP but failed to pursue a 2nd application.
Some students selected U and V' in such a way that the powers of x increased, which made the solution more difficult to complete.

$$\therefore I_n = -\frac{\cos 1}{(2n-1)!} + \left(\frac{\sin 1}{(2n-2)!} - I_{n-1}\right)$$

$$\text{i.e.}\quad I_n = \frac{\sin 1}{(2n-2)!} - \frac{\cos 1}{(2n-1)!} - I_{n-1}$$

(c) (ii) ❶ Prove true for $n = 1$.

3 marks

- 1st mark for correct execution of step 1 (proving true for $n = 1$).
- 2nd mark for first few lines of proof in step 3.

$$I_1 = \int_0^1 \frac{x^{2\times 1-1}}{(2\times 1-1)!}\sin x\,dx$$

$$= \int_0^1 x\sin x\,dx$$

Let $u = x \quad v' = \sin x$

$u' = 1 \quad v = -\cos x$

$$I_1 = [x \times -\cos x]_0^1 - \int_0^1 1 \times -\cos x\,dx$$

$$= -\cos 1 - 0 + [\sin x]_0^1$$

$$= -\cos 1 + \sin 1 - 0$$

$$= \frac{1\times\sin 1 + (-1)\times\cos 1}{(2\times 1-1)!}$$

$$= \frac{a_1\sin 1 + b_1\cos 1}{(2\times 1-1)!}$$

where $a_1 = 1, b_1 = -1$

$\therefore$ Statement is true for $n = 1$.

❷ Assume statement is true for $n = k$.

i.e. assume $I_k = \dfrac{a_k\sin 1 + b_k\cos 1}{(2k-1)!}$ where $a_k, b_k \in \mathbb{Z}$.

❸ Prove statement is true for $n = k + 1$.

i.e. prove $I_{k+1} = \dfrac{a_{k+1}\sin 1 + b_{k+1}\cos 1}{(2(k+1)-1)!}$ where $a_k, b_k \in \mathbb{Z}$.

$$LHS = I_{k+1}$$

$$= \frac{\sin 1}{(2(k+1)-2)!} - \frac{\cos 1}{(2(k+1)-1)!} - I_{k+1-1}$$

$$= \frac{\sin 1}{(2k+2-2)!} - \frac{\cos 1}{(2k+2-1)!} - I_k$$

$$= \frac{\sin 1}{(2k)!} - \frac{\cos 1}{(2k+1)!} - \frac{a_k \sin 1 + b_k \cos 1}{(2k-1)!}$$

$$= \frac{(2k+1)\sin 1}{(2k+1)(2k)!} - \frac{\cos 1}{(2k+1)!} - \frac{(2k+1)\times 2k \times (a_k \sin 1 + b_k \cos 1)}{(2k+1)\times 2k \times (2k-1)!}$$

$$= \frac{(2k+1)\sin 1 - \cos 1 - 2k(2k+1)(a_k \sin 1 + b_k \cos 1)}{(2k+1)!}$$

$$= \frac{\big((2k+1) - 2k(2k+1)a_k\big)\sin 1 + \big(-1 - 2k(2k+1)b_k\big)\cos 1}{(2(k+1)-1)!}$$

$$= \frac{a_{k+1}\sin 1 + b_{k+1}\cos 1}{(2(k+1)-1)!}$$

$\therefore LHS = RHS$

where:

$a_{k+1} = (2k+1) - 2k(2k+1)a_k$

$b_{k+1} = -1 - 2k(2k+1)b_k$

❹ If true for $n = k$, then true for $n = k + 1$.

Since proven true for $n = 1$, statement is proven true by mathematical induction for every positive integer n.

Comment:

Well done by many students, though many solutions failed to correctly establish the result in step 1 (for $n = 1$). Whilst there was some acceptance for students who did prove true for $n = 2$, there was a requirement that the integral for I_1 was evaluated (by integration by parts) as part of the step 1 process. A range of students had some difficulties with correct factorial expressions (and associated simplifying) and in particular, the recursive relationship for factorials e.g. $(2n-1)! = (2n-1)\times(2n-2)!$.

The best solutions explicitly referred to the form of the expression of interest (including reference to a_k, b_k, a_1, b_1 and the denominator in the appropriate form) throughout the key stages of proof within their solution.

(c) (iii) Since $0 < I_n < \dfrac{1}{(2n)!}$

2 marks

1st mark for progressing to the step marked as ⊂.

$$0 < \frac{a_n \sin 1 + b_n \cos 1}{(2n-1)!} < \frac{1}{(2n)!}$$

Multiply through by $(2n-1)!$

$$0 < a_n \sin 1 + b_n \cos 1 < \frac{1}{2n \times (2n-1)!} \times (2n-1)!$$

$$0 < a_n \sin 1 + b_n \cos 1 < \frac{1}{2n}$$

Divide through by $\cos 1$ and considering $\cos 1 > \dfrac{1}{2} > 0$.

$$0 < \frac{a_n \sin 1}{\cos 1} + \frac{b_n \cos 1}{\cos 1} < \frac{1}{2n} \times \frac{1}{\cos 1} \quad ⊂$$

$$0 < a_n \tan 1 + b_n < \frac{1}{2n} \times \frac{1}{\cos 1}$$

Since $\cos 1 > \dfrac{1}{2}, \dfrac{1}{\cos 1} < 2$ and $\dfrac{1}{2n} \times \dfrac{1}{\cos 1} < \dfrac{1}{2n} \times 2.$ ☹

Hence:

$$0 < a_n \tan 1 + b_n < \frac{1}{2n} \times \frac{1}{\cos 1} < \frac{1}{2n} \times 2.$$

$\therefore\ 0 < a_n \tan 1 + b_n < \dfrac{1}{n}$

Comment:

Not well executed, even by a range of students who may have received full marks. The level of clarity was questionable for many solutions, with concerns about the clarity of justifying the result:

$$\frac{1}{2n} \times \frac{1}{\cos 1} < \frac{1}{2n} \times 2$$

This is a key relationship that should have been explicitly communicated within the solution and very few students really produced a quality response that nailed this in a satisfying way. The ☹ in the solution reflects the vibe of the marker trying to make sense of many solutions and how they were being justified.

(c) (iv) Assume $\tan 1$ is rational, i.e. assume $\tan 1 = \dfrac{p}{q}$

where $p, q \in \mathbb{Z}^+$ and $HCF(p, q) = 1$.

Using part (iii):

$$0 < a_n \times \frac{p}{q} + b_n < \frac{1}{n} \quad ◎$$

As this inequality is true for all positive integer values of n, then it will be true for $n = q$ since q is a positive integer.

Hence, using $n = q$:

$$0 < a_q \times \frac{p}{q} + b_q < \frac{1}{q}$$

Multiplying through by q, where $q > 0$:

$$0 < a_q \times p + b_q \times q < 1$$

Given that a_q, b_q, p and q are all integers, the value of the expression $a_q \times p + b_q \times q$ must also be an integer.

Hence, it cannot have a value between 0 and 1 as stated in .

$\therefore$ The assumption that $\tan 1$ is rational must be false.

Hence, by contradiction, $\tan 1$ is irrational.

<u>2 marks</u>

1st mark for progressing to the step marked as ◎.

<u>Comment:</u>

No student received full marks for the question. Many students received the first mark.

Stating the assumption that $\tan 1 = \frac{p}{q}$ was not sufficient for the first mark and students were expected to incorporate the result from part (iii) as it is explicitly requested in the question.